

# MULTIVARIABLE HERMITE POLYNOMIALS AND PHASE-SPACE DYNAMICS

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## Abstract

The phase-space approach to classical and quantum systems demands for advanced analytical tools. Such an approach characterizes the evolution of a physical system through a set of variables, reducing to the canonically conjugate variables in the classical limit. It often happens that phase-space distributions can be written in terms of quadratic forms involving the above quoted variables. A significant analytical tool to treat these problems may come from the generalized many-variables Hermite polynomials, defined on quadratic forms in  $\mathcal{R}^n$ . They form an orthonormal system in many dimensions and seem the natural tool to treat the harmonic oscillator dynamics in phase-space. In this contribution we discuss the properties of these polynomials and present some applications to physical problems.

## 1 Introduction

Classical special functions play a central role in both pure and applied mathematics. Usually conceived for the solution of very specific problems, they gave rise to a far-reaching theory, being part of, and frequently motivation for, important general theories.

The trigonometric functions, for instance, originally introduced to deal with specific problems of astronomy and navigation, are the basis for the theory of Fourier series and Fourier integral, which have applications to many parts of physics.

Similarly, Bessel functions firstly appeared in mathematical physics in the 1738 Bernoulli's memoir, containing enunciations of theorems on the oscillations of heavy chains. Then, they reappeared in mechanical problems as the vibration of a stretched membrane or the symmetrical and unsymmetrical propagation of heat in solid cylinders and spheres, as well as in astronomical problems, related for instance to the elliptic motion of a planet about the sun. Presently, Bessel functions have a very wide field of applications, from abstract number theory and theoretical astronomy to concrete problems of physics and engineering [1].

Correspondingly, classical orthogonal polynomials (Jacobi, Legendre, Hermite), also introduced in connection with astronomical problems, are now of great importance in mathematical physics, approximation theory as well as in the theory of mechanical quadrature. In addition, they find significant applications in quantum mechanics on the determination of discrete energy spectra and the corresponding wave functions in fundamental problems [2]. Orthogonal polynomials are indeed essential tool to deal with the problems of the harmonic oscillator and the motion of particles in a central field. Furthermore, the classical orthogonal polynomials of a discrete variable are of interest in the theory of difference methods [2].

Theory of classical special functions is rather well settled [3]. Recursion relations, addition theorems, integral representations, generating functions, asymptotic formulae, differential equations are collected in an organic body, which is however continuously refined and enriched by new investigations and new theoretical approaches [4].

Let us recall for instance the method illustrated in ref. [1], which suggests a generalization of the Rodriguez formula for the classical orthogonal polynomials, thus allowing to obtain explicit integral representations of all the special functions and to derive their basic properties. Similarly, the possibility of framing special functions within the context of group theory [4, 5] revealed a powerful tool permitting derivation of new results and a rational classification of old results, as well as suggesting to introduce new classes of functions, related to the recently discovered algebraic systems, such as the supergroups and the quantum groups [6].

Furthermore, we note the description of orthogonal polynomials by their recursion relations, which once regarded as eigenvalue equations allow to look at orthogonal polynomials from the viewpoint of scattering theory [4, 7].

Recently, interest in special functions has greatly increased in connection with the possibility of *generalizing* the well known functions of mathematical physics to more than one variable and/or more than one index. In this regard, the generalization amounts to introducing functions with properties analogous to those of the one-variable counterpart. Generating functions are usually the key-note for many-variable generalizations of special functions.

The multivariable Bessel functions, for instance, originally introduced by Appell [8] in connection with the problem of the elliptic motion of planets [9], have revealed a wealth of possible applications to physical and/or purely mathematical problems, as the scattering of laser radiation by free or weakly bounded electrons, the emission of e.m. radiation by relativistic electrons passing through magnetic undulators [10] as well as problems related to the queuing theory [11]. Also, they proved their relevance in multiphoton emission and absorption processes by quantum systems, which are of interest for the investigation of squeezed states, the relevant Hamiltonian operator containing indeed powers of the annihilation and creation operators [12].

Correspondingly, the multivariable generalization of orthogonal polynomials has attracted a great amount of interest. In particular, as to the Hermite polynomials, let us recall that in ref. [13] a procedure has been developed, which generalizing that proposed by Gould and Hopper [14] allows to define multivariable generalized Hermite polynomials, providing a complete orthonormal set in  $\mathcal{L}_2(R^n)$  space of square sommable functions with  $n$  variables.

The present paper concerns with the classical many-variable functions introduced by Hermite [15], whose application within the context of the phase space approach to physical problems is suggested. Accordingly, in Sec. 2 a general view on the many-variable Hermite polynomials is presented. The possibility of exploiting the developed formalism within the context of the phase

space picture, which is becoming the context where many physical problems are naturally framed, is investigated in Sec. 3. In particular, the basic model of the harmonic oscillator is considered. General considerations and extensions are presented in Sec. 4.

## 2 Many-variable Hermite polynomials

One-variable Hermite polynomials  $\mathcal{H}_n(x)$ <sup>1</sup> can be defined by means of the relation

$$e^{-\frac{1}{2}(x-t)^2} = e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{H}_n(x) , \quad (1)$$

where they appear as the coefficients of the series expansion of the exponential of a quadratic form defined on the real domain.

The above expression can also be recast in the more usual form

$$e^{xt-t^2/2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{H}_n(x) , \quad (2)$$

from which, exploiting the series development of the exponential function and appropriately rearranging the summation, the well known expression of the Hermite polynomials in form of a finite sum can be easily drawn:

$$\mathcal{H}_n(x) = n! \sum_{l=0}^{[n/2]} \frac{(-)^l x^{n-2l}}{l! 2^l (n-2l)!} , \quad (3)$$

where  $[v]$  denotes the largest integer  $\leq v$ .

Also, according to the formula for the Taylor expansion, (2) provides the Rodriguez formula:

$$\mathcal{H}_n(x) = (-)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} . \quad (4)$$

Taking the derivatives of both sides of (1) or (2) with respect to  $x$  and  $t$ , it is easy to infer the recursion relations

$$\begin{aligned} \mathcal{H}_{n+1} &= x\mathcal{H}_n - n\mathcal{H}_{n-1} , \\ \mathcal{H}'_n &= n\mathcal{H}_{n-1} , \end{aligned} \quad (5)$$

linking the polynomial of order  $n$  to the contiguous ones. The prime denotes derivative with respect to  $x$ .

It is immediate to get from the above relations the differential equation obeyed by  $\mathcal{H}_n$ 's:

$$\left( -\frac{d^2}{dx^2} + x \frac{d}{dx} \right) \mathcal{H}_n = n\mathcal{H}_n , \quad (6)$$

which allows to understand  $\mathcal{H}_n$  as eigenfunction of the operator

$$\hat{\mathcal{D}} = -\frac{d^2}{dx^2} + x \frac{d}{dx} , \quad (7)$$

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<sup>1</sup>The use of the script  $\mathcal{H}_n$  to denote the Hermite polynomial is in order to avoid confusion with the more common polynomial  $H_n$  :  $H_n(x) = 2^{n/2} \mathcal{H}_n(\sqrt{2}x)$ .

$n$  being the corresponding eigenvalue.

The operator  $\hat{\mathcal{D}}$  is not self adjoint; the adjoint operator

$$\hat{\mathcal{D}}^+ = -\frac{d^2}{dx^2} - x\frac{d}{dx} - 1 \quad (8)$$

admits the eigenfunction  $\mathcal{H}_n^+$ , explicitly given as

$$\mathcal{H}_n^+(x) = \mathcal{H}_n(x)e^{-x^2/2} \quad (9)$$

belonging to the same eigenvalue as  $\mathcal{H}_n$ . Accordingly, the functions  $(\mathcal{H}_n, \mathcal{H}_n^+)$  form a biorthogonal set in the usual sense that they satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} dx \mathcal{H}_n(x) \mathcal{H}_m(x) e^{-x^2/2} = n! \sqrt{2\pi} \delta_{nm} . \quad (10)$$

Generalizing the relation (1) to involve a bilinear form defined in  $R^n$ , Hermite introduced many-variable functions [15]. Adopting a matrix notation, we can write down

$$e^{-\frac{1}{2}(\underline{x}-\underline{h})^T \underline{M}(\underline{x}-\underline{h})} = e^{-\frac{1}{2}\underline{x}^T \underline{M} \underline{x}} \sum_{m_1, \dots, m_n} \frac{h_1^{m_1}}{m_1!} \dots \frac{h_n^{m_n}}{m_n!} \mathcal{H}_{m_1, \dots, m_n}(\underline{x}) , \quad (11)$$

where  $\underline{x}$  and  $\underline{h}$  are elements of the vector space  $R^n$ :

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} , \quad \underline{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} , \quad (12)$$

the superscript  $T$  meaning transpose.

Accordingly,  $\underline{M}$  is a real  $n \times n$  matrix:  $\underline{M} = (a_{ij})$ ,  $i, j = 1, \dots, n$ , which is required to be symmetric:  $a_{ij} = a_{ji}$ , not degenerate, i.e.  $\det \underline{M} \geq 0$  and positive definite:  $a_{ii} > 0$ ,  $i = 1, \dots, n$ .

The expression (11) can be rewritten in the alternative form

$$e^{\underline{x}^T \underline{M} \underline{h} - \frac{1}{2}\underline{h}^T \underline{M} \underline{h}} = \sum_{m_1, \dots, m_n} \frac{h_1^{m_1}}{m_1!} \dots \frac{h_n^{m_n}}{m_n!} \mathcal{H}_{m_1, \dots, m_n}(\underline{x}) , \quad (13)$$

which is the  $n$ -variable analog of (2).

In passing, it is worth noticing that the above expression suggests considering the more general bilinear form

$$\phi(\underline{x}, \underline{h}) = \underline{x}^T \underline{A} \underline{h} - \frac{1}{2} \underline{h}^T \underline{B} \underline{h} \quad (14)$$

with the matrices  $\underline{A}$ ,  $\underline{B}$  being symmetric but in general different from each other. In this respect, let us recall that the Grassman Hermite functions have been introduced using the above quadratic form, with  $\underline{A}$  and  $\underline{B}$  being antisymmetric matrices and  $\underline{x}$ ,  $\underline{h}$  anticommuting variables. A further extension of (14) have been considered in ref. [6], with the intent of obtaining a class of functions, related to quantum groups.

In full analogy with the one-variable case, taking the derivative of (11) or (13) with respect to the components of both vectors  $\underline{x}$  and  $\underline{h}$ , we get the recursion relations satisfied by  $\mathcal{H}_{m_1, \dots, m_n}(\underline{x})$ :

$$\begin{aligned} \mathcal{H}_{m_1, \dots, m_1+1, \dots, m_n}(\underline{x}) &= \left( \sum_{j=1}^n a_{ij} x_j \right) \mathcal{H}_{m_1, \dots, m_n}(\underline{x}) - \sum_{j=1}^n a_{ij} m_j \mathcal{H}_{m_1, \dots, m_j-1, \dots, m_n}(\underline{x}) , \\ \frac{\partial}{\partial x_i} \mathcal{H}_{m_1, \dots, m_n}(\underline{x}) &= \sum_{j=1}^n a_{ij} m_j \mathcal{H}_{m_1, \dots, m_j-1, \dots, m_n}(\underline{x}) , \end{aligned} \quad (15)$$

with  $i$  ranging from 1 to  $n$ .

Since corresponding to any quadratic form  $\phi(\underline{x}) = \underline{x}^T \underline{M} \underline{x}$  we can associate the adjoint form defined in terms of the inverse matrix, i.e.  $\psi(\underline{\xi}) = \underline{\xi} \underline{M}^{-1} \underline{\xi}$ , it is possible to associate with the  $\mathcal{H}_{m_1, \dots, m_n}$ 's the adjoint polynomials  $\mathcal{G}_{m_1, \dots, m_n}(\underline{x})$ . The definition resembles the relation (11), but involves the transformed vectors:

$$\underline{\xi} \equiv \underline{M} \underline{x} , \quad \underline{k} \equiv \underline{M} \underline{h} . \quad (16)$$

Explicitly, we have indeed

$$e^{-\frac{1}{2}(\underline{\xi}-\underline{k})^T \underline{M}^{-1}(\underline{\xi}-\underline{k})} = e^{-\frac{1}{2}\underline{\xi}^T \underline{M}^{-1} \underline{\xi}} \sum_{m_1, \dots, m_n} \frac{k_1^{m_1}}{m_1!} \cdots \frac{k_n^{m_n}}{m_n!} \mathcal{G}_{m_1, \dots, m_n}(\underline{x}) . \quad (17)$$

If  $\underline{M}$  is the identity matrix, the two polynomials coincide:  $\mathcal{H}_{m_1, \dots, m_n} = \mathcal{G}_{m_1, \dots, m_n}$ . Furthermore, they turn into the product of  $n$  one-variable Hermite polynomials, i.e.

$$\mathcal{H}_{m_1, \dots, m_n}(\underline{x}) = \prod_{i=1}^n \mathcal{H}_{m_i}(x_i) . \quad (18)$$

Taking the derivative of (17) with respect to  $\underline{x}$  and  $\underline{k}$ , the following set of recursion relations can be obtained:

$$\begin{aligned} \mathcal{G}_{q_1, \dots, q_i+1, \dots, q_n}(\underline{x}) &= x_i \mathcal{G}_{q_1, \dots, q_n} - \frac{1}{\Delta} \sum_{j=i}^n A_{ij} q_j \mathcal{G}_{q_1, \dots, q_j-1, \dots, q_n}(\underline{x}) , \\ \frac{\partial}{\partial x_i} \mathcal{G}_{q_1, \dots, q_n}(\underline{x}) &= q_i \mathcal{G}_{q_1, \dots, q_i-1, \dots, q_n}(\underline{x}) , \end{aligned} \quad (19)$$

with  $\Delta \equiv \det \underline{M}$  and  $A_{ij}$  the minor relevant to the element  $a_{ij}$ .

The relevance of the polynomials  $\mathcal{G}_{q_1, \dots, q_n}$  is clarified by the orthogonality relation, which can be proved in the form

$$\int_{R^n} d\underline{x} e^{-\frac{1}{2}\underline{x}^T \underline{M} \underline{x}} \mathcal{H}_{m_1, \dots, m_n}(\underline{x}) \mathcal{G}_{q_1, \dots, q_n}(\underline{x}) = \frac{1}{\sqrt{\Delta}} \prod_{i=1}^n m_i! \sqrt{2\pi} \delta_{m_i, q_i} . \quad (20)$$

For the explicit derivation of the above relation the reader is addressed to ref. [8].

The orthogonality relation (20) can be conveniently exploited to express a given  $n$ -variable function  $\rho(\underline{x})$  in form of a series involving  $\mathcal{H}_{m_1, \dots, m_n}$  and  $\mathcal{G}_{m_1, \dots, m_n}$ . Accordingly, let us put

$$\rho(\underline{x}) = \sum_{m_1, \dots, m_n} A_{m_1, \dots, m_n} \mathcal{H}_{m_1, \dots, m_n}(\underline{x}) , \quad (21)$$

or

$$\rho(\underline{x}) = \sum_{m_1, \dots, m_n} B_{m_1, \dots, m_n} \mathcal{G}_{m_1, \dots, m_n}(\underline{x}) , \quad (22)$$

with the coefficients  $A_{m_1, \dots, m_n}$ ,  $B_{m_1, \dots, m_n}$  being specialized according to (20) into

$$\begin{aligned} A_{m_1, \dots, m_n} &= \frac{\sqrt{\Delta}}{(2\pi)^{n/2}} \frac{1}{m_1!} \cdots \frac{1}{m_n!} \int_{R^n} d\underline{x} e^{-\frac{1}{2}\underline{x}^T \underline{M} \underline{x}} \rho(\underline{x}) \mathcal{G}_{m_1, \dots, m_n}(\underline{x}) , \\ B_{m_1, \dots, m_n} &= \frac{\sqrt{\Delta}}{(2\pi)^{n/2}} \frac{1}{m_1!} \cdots \frac{1}{m_n!} \int_{R^n} d\underline{x} e^{-\frac{1}{2}\underline{x}^T \underline{M} \underline{x}} \rho(\underline{x}) \mathcal{H}_{m_1, \dots, m_n}(\underline{x}) . \end{aligned} \quad (23)$$

The explicit values of the entries of the matrix  $\underline{M}$  should be suggested by the specific problem under study.

It is needless to say that the theory of many-variable Hermite polynomials is very rich. However, the above considerations represent all the machinery we need for testing the possibility of using these functions as basis for the phase-space analysis of physical problems. For a more detailed discussion the interested reader is addressed to refs. [8, 15].

### 3 Two-variable Hermite polynomials and phase space picture of dynamical problems

Phase space picture is becoming the unifying language for both classical and quantum mechanics. Phase space formalism is indeed basic to the Hamiltonian formulation of classical mechanics. In this connection, the evolution of a dynamical system is described by a number  $n$  of independent coordinate variables and on the same number of canonically conjugate momenta. The cartesian space of these  $2n$  coordinates is just the *phase-space*.

Correspondingly, phase space picture of quantum mechanics is becoming increasingly popular. Although, the concept of phase-space is not compatible with quantum mechanics,  $\hat{q}$  and  $\hat{p}$  being noncommuting operators, the Wigner phase-space representation allows to overcome this problem, since in this representation both the coordinate and momentum variables are  $c$ -numbers. Accordingly, it is possible to perform phase-space canonical transformations as in the case of classical mechanics, which correspond to unitary transformations in the Schrödinger picture of quantum mechanics.

Phase space concept appears therefore as the unifying context, where classical as well as quantum mechanics can be naturally framed, thus suggesting the possibility to transfer concepts and methods from quantum to classical mechanics and viceversa.

Furthermore, as discussed in ref. [16], phase-space picture provides the natural language for quantum optics as well, offering a geometrical view to coherent and squeezed states as circles and ellipses respectively. In this connection, taking advantages from the symmetry of the relevant Wigner phase space distribution function it is possible to calculate expectation values and transition probabilities for the above quoted states [16].

Finally, let us recall that phase space is the context where the dynamics of electron beams moving through magnetic channels is studied and the Hamiltonian optics can be conveniently reformulated.

As already remarked, the paper is aimed at investigating the possibility of using the many-variable Hermite polynomials, briefly discussed in the previous section, as analytical tool in the phase-space approach to dynamical problems.

In the quantum framework, one-variable Hermite polynomials are intimately related to the harmonic oscillator dynamics. It is therefore natural to analyse, as first step for our investigation, the harmonic oscillator dynamics in the phase-space representation.

Let us consider therefore the quadratic Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}k(s)q^2, \quad (24)$$

the variable  $s$  playing the role of *time*. It is needless to stress the relevance of the above Hamiltonian as basic model for many physical problems as well as approximation in many of the existing theories.

As already stressed, in quantum mechanics the Hamiltonian (24) rules the evolution of a harmonic oscillator of unit mass and *time*-dependent frequency  $k(s)$ ,  $\hat{q}$  and  $\hat{p}$  being the position and momentum operators. In classical mechanics, it describes for instance the betatron motion of a charged particle through a magnetic quadrupole, as in the ray optics it governs the propagation of an optical ray through a nonhomogeneous medium with a quadratic profile of the refractive index.

It is interesting to notice that in the configuration space picture the evolution of the system described by the Hamiltonian (24) is analysed within the context of conceptually and formally different approaches: the Schrödinger equation for the quantum wave function and the Hamilton equation of motion for the canonically conjugate variables  $q$  and  $p$ . Conversely, the Von Neumann equation for the Wigner distribution function and the classical Liouville equation for the phase-space distribution are of the same form, so long as the Hamiltonian of the quantum system is quadratic. Hence, time evolution of the Wigner function can be obtained directly from the solution of the equation of motion of the corresponding classical system. Harmonic oscillator provides a unique example in which classical and quantum phenomenology overlap to a large extent.

Let us approach the problem within the context of classical mechanics. Accordingly, the Liouville equation for the phase space distribution function  $\rho(q, p; s)$  is immediately written down as

$$\frac{\partial}{\partial s}\rho(q, p; s) = \left\{ -p\frac{\partial}{\partial q} + k(s)q\frac{\partial}{\partial p} \right\} \rho(q, p; s), \quad (25)$$

with an assigned initial condition:  $\rho(q, p; s) = \rho_0(q, p)$ .

As introduction to the forthcoming discussion, let us recall that an invariant quadratic form

$$I = \underline{x}^T \underline{T}(s) \underline{x} \quad (26)$$

can be associated to any dynamics described by quadratic Hamiltonians in canonical coordinates and momenta. In passing, it is worth stressing that within a quantum context the quadratic form (26) is reported as the Ermakov–Lewis invariant [17], the vector  $\underline{x}$  containing obviously the position and momentum operators, whilst in classical mechanics it is reported as the Courant–Snyder invariant [18], firstly introduced in the analysis of electron beam motion through magnetic channels. In the above expression, the two component vector  $\underline{x} \equiv \begin{pmatrix} q \\ p \end{pmatrix}$  is acted by the real  $2 \times 2$

matrix  $\underline{T}$ , which furthermore is required to be symmetric and unimodular:  $\det \underline{T} = 1$ . Just to share the language of accelerator physics, we refer to  $\underline{T}$  as the *Twiss* matrix<sup>2</sup> and report it in the form

$$\underline{T} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} . \quad (27)$$

The entries  $\alpha, \beta, \gamma$ , usually named as *Twiss* parameters, play an important role in designing transport channels.

The quadratic form (26) can be depicted in the phase space as an ellipse, whose size and orientation are determined by the Twiss coefficients. The area of the ellipse, which is just the value of the invariant  $I$ , is usually denoted in accelerator physics as  $I = \pi\epsilon$ ,  $\epsilon$  being named as the *beam emittance*. It plays a crucial role in characterizing the quality and the dynamics of the  $e$ -beam. In a single particle analysis,  $\alpha, \beta, \gamma$  defines the contour of the particle trajectory, as in an ensemble analysis they define the second order momentum of the phase space distribution function, thus providing information on its extent and maximum localization. Explicitly,

$$\begin{aligned} \epsilon\gamma &= \sigma_{pp}^2 = \langle p^2 \rangle - \langle p \rangle^2 , \\ \epsilon\beta &= \sigma_{qq}^2 = \langle q^2 \rangle - \langle q \rangle^2 , \\ \epsilon\alpha &= -\sigma_{qp}^2 = -(\langle qp \rangle - \langle q \rangle \langle p \rangle) , \end{aligned} \quad (28)$$

the averages being understood on the distribution function. Accordingly, the emittance  $\epsilon$  can be given the further meaning:

$$\epsilon^2 = \sigma_{qq}^2 \sigma_{pp}^2 - \sigma_{qp}^2 . \quad (29)$$

Let us stress now that the Liouville equation admits as particular solution the distribution function

$$\rho(q, p; s) = \frac{1}{2\pi\epsilon} \exp \left\{ -\frac{1}{2\epsilon} \underline{x}^T \underline{T} \underline{x} \right\} , \quad (30)$$

shaped indeed in form of a Gaussian and therefore with a maximum localization within the ellipse of area  $\epsilon$ .

The above considerations suggest to use the functions  $\mathcal{H}_{m,n} e^{-\frac{1}{2\epsilon} \underline{x}^T \underline{T} \underline{x}}$  as basis in the phase-plane, the entries of the matrix  $\underline{T}$  being chosen as the second order momentum of the distribution. In other words we can use the above quoted functions to approximate a generic function defined in the phase space by means of two-variable Gaussians, further modelled by the Hermite polynomials  $\mathcal{H}_{m,n}$ , which give a different maximum localization. The process is perfectly similar to that used for one-variable functions using the Hermite functions.

Let us consider therefore generic distribution function  $\rho(q, p; s)$ . According to the results of Sec. 2, we can express  $\rho(q, p; s)$  in form of a series:

$$\rho(q, p; s) = \sum_{m,n} a_{m,n}(s) \rho_{m,n}^H(q, p) , \quad (31)$$

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<sup>2</sup>The quadratic form (26) can also be regarded as the transcription in phase space of the quantum invariant, the vector  $\underline{x}$  being formed by the expectation values of the position and momentum operators and the matrix  $\underline{T}$  being linked to the *covariance* matrix.

with  $a_{m,n}(s)$  and  $\rho_{m,n}^H(q, p)$  being explicitly given as

$$\begin{aligned} a_{m,n}(s) &= \int dq \int dp \rho(q, p; s) \mathcal{G}_{m,n}(q, p) , \\ \rho_{m,n}^H(q, p) &= \frac{1}{2\pi\epsilon m!n!} \mathcal{H}_{m,n}(q, p) e^{-\frac{1}{2\epsilon} \underline{x}^T \underline{T} \underline{x}} . \end{aligned} \quad (32)$$

The superscript  $H$  specifies that the polynomials  $\mathcal{H}_{m,n}$  have been used in the series expansion.

The alternative expansion in terms of the adjoint polynomials  $\mathcal{G}_{m,n}(q, p)$  can be also used, thus leading to

$$\rho(q, p; s) = \sum_{m,n} b_{m,n} \rho_{m,n}^G(q, p) , \quad (33)$$

with

$$\begin{aligned} b_{m,n}(s) &= \int dq \int dp \rho(q, p; s) \mathcal{H}_{m,n}(q, p) , \\ \rho_{m,n}^G(q, p) &= \frac{1}{2\pi\epsilon m!n!} \mathcal{G}_{m,n}(q, p) e^{-\frac{1}{2\epsilon} \underline{x}^T \underline{T} \underline{x}} , \end{aligned} \quad (34)$$

and the superscript  $G$  specifying that the adjoint polynomials  $\mathcal{G}_{m,n}$  have been used.

The entries of the *Twiss* matrix appearing in the above expression can be conveniently chosen, according to the previous discussion, as the second-order momenta of the distribution function  $\rho_0(q, p)$  at the initial *time*. Then, the *emittance*  $\epsilon$  is just obtained according to (29).

Inserting the expression (31) into the Liouville equation (25), we get the set of equations for the coefficients  $a_{m,n}(s)$ , namely

$$\begin{aligned} \frac{d}{ds} a_{m,n} &= (\sigma_{pp}^2 - k\sigma_{qq}^2) m n a_{m-1,n-1} + \sigma_{qp} [m(m-1) a_{m-2,n} - k n(n-1) a_{m,n-2}] \\ &+ m a_{m-1,n+1} - k n a_{m+1,n-1} , \end{aligned} \quad (35)$$

where the relations

$$\frac{\partial}{\partial q} \rho_{m,n}^H = -(m+1) \rho_{m+1,n}^H , \quad \frac{\partial}{\partial p} \rho_{m,n}^H = -(n+1) \rho_{m,n+1}^H \quad (36)$$

and

$$\begin{aligned} p \rho_{m,n}^H &= \rho_{m,n-1}^H + \sigma_{qp} (m+1) \rho_{m+1,n}^H + \sigma_{qq}^2 (n+1) \rho_{m,n+1}^H , \\ q \rho_{m,n}^H &= \rho_{m-1,n}^H + \sigma_{pp}^2 (m+1) \rho_{m+1,n}^H + \sigma_{qp}^2 (n+1) \rho_{m,n+1}^H \end{aligned} \quad (37)$$

have been used, obtained from the recursive relations obeyed by  $\mathcal{H}_{m,n}$  (see Appendix A).

Let us consider for instance the particular case where the quadrupole strength does not change along the direction of motion:  $k(s) = k$ ; and the initial distribution function  $\rho_0(q, p)$  has the form

$$\rho_0(q, p) = \frac{1}{2\pi\epsilon} \exp \left\{ -\frac{1}{2\epsilon} \underline{x}^T \underline{T} \underline{x} \right\} . \quad (38)$$

The coefficients  $a_{m,n}(0)$  are then given as

$$a_{m,n}(0) = \delta_{m,0} \delta_{n,0} . \quad (39)$$

In addition, assuming that the electron beam is matched to the quadrupole, namely

$$\sigma_{qp} = 0 \ , \quad \sigma_{pp}^2 - k\sigma_{qq}^2 = 0 \ , \quad (40)$$

it is easy to prove that the distribution function does not change during the motion:

$$\rho(q, p; s) = \rho_0(q, p) \ . \quad (41)$$

More general situations require the numerical handling of eq. (35), from which the evolution of  $\rho(q, p; s)$  can be inferred according to the expansion (31).

## 4 Concluding remarks

The analysis developed in the previous sections has been aimed at testing the possibility of using many-variable Hermite polynomials as analytical tool, within the context of phase-space picture to dynamical problems.

The discussion has been limited, for illustrative purposes, to the harmonic oscillator dynamics. However, it can be easily realized that Hamiltonians containing higher order terms, accounting for non linear forces, can be treated by means of the same formalism as well.

Also, within a quantum mechanical context, the formalism developed might be used in connection with the Von Neumann equation, which, as already noticed, rules the evolution of the Wigner distribution function  $W(q, p; s)$  according to [19]

$$\frac{\partial}{\partial s} W(q, p; s) = \left\{ -p \frac{\partial}{\partial q} + \frac{1}{i\hbar} \left[ V \left( q + \frac{1}{2} i\hbar \frac{\partial}{\partial p} \right) - V \left( q - \frac{1}{2} i\hbar \frac{\partial}{\partial p} \right) \right] \right\} W(q, p; s) \ . \quad (42)$$

Furthermore, let us say that, although in Sec. 3 we have considered 1-dimensional dynamics, problems with more than one degree of freedom can be analysed, as, for instance, the motion of electron-beams along magnetic channels with transverse coupling. In that case, the radial and vertical motions cannot be separated and the dynamics should be analysed in a 4-dimensional phase-space, the relevant elements consisting of the conjugate variables  $x, p_x, y, p_y$ .

## APPENDIX A

### Two-variable Hermite polynomials

This Appendix is devoted to discuss with some details the properties of the two-variable Hermite polynomials, which are used in Sec. 3 as illustrative of the usefulness of such functions within the context of the phase-space formalism.

Firstly, let us write down the explicit form of the recursion relations, reported in (15) for the general case of many-variable polynomials:

$$\begin{aligned}
 \mathcal{H}_{m+1,n}(q,p) &= (aq + bp)\mathcal{H}_{m,n}(q,p) - am\mathcal{H}_{m-1,n}(q,p) - bn\mathcal{H}_{m,n-1}(q,p) , \\
 \mathcal{H}_{m,n+1}(q,p) &= (bq + cp)\mathcal{H}_{m,n}(q,p) - bm\mathcal{H}_{m-1,n}(q,p) - cn\mathcal{H}_{m,n-1}(q,p) , \\
 \frac{\partial}{\partial q}\mathcal{H}_{m,n}(q,p) &= am\mathcal{H}_{m-1,n}(q,p) + bn\mathcal{H}_{m,n-1}(q,p) , \\
 \frac{\partial}{\partial p}\mathcal{H}_{m,n}(q,p) &= bm\mathcal{H}_{m-1,n}(q,p) + cn\mathcal{H}_{m,n-1}(q,p) .
 \end{aligned} \tag{A.1}$$

From the above relations after some algebra the following partial differential equation can be deduced

$$\left[ -c\frac{\partial^2}{\partial q^2} - a\frac{\partial^2}{\partial p^2} + 2b\frac{\partial^2}{\partial q\partial p} + \Delta \left( q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p} \right) \right] \mathcal{H}_{m,n} = \Delta(m+n)\mathcal{H}_{m,n} , \tag{A.2}$$

in some sense reminiscent of eq. (6), which can be recovered in correspondence with  $\underline{\underline{M}} = \underline{\underline{I}}$ .

Consequently, the polynomials  $\mathcal{H}_{m,n}$  can be understood as eigenfunctions of the operator

$$\hat{T} = -c\frac{\partial^2}{\partial q^2} - a\frac{\partial^2}{\partial p^2} + 2b\frac{\partial^2}{\partial q\partial p} + \Delta \left( q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p} \right) , \tag{A.3}$$

with eigenvalue  $\Delta(m+n)$ .

Let us consider now the adjoint polynomials  $\mathcal{G}_{m,n}(q,p)$ , for which the general relations (19) specialize into

$$\begin{aligned}
 \mathcal{G}_{m+1,n}(q,p) &= q\mathcal{G}_{m,n}(q,p) - cm\mathcal{G}_{m-1,n}(q,p) + bn\mathcal{G}_{m,n-1}(q,p) , \\
 \mathcal{G}_{m,n+1}(q,p) &= p\mathcal{G}_{m,n}(q,p) + bm\mathcal{G}_{m-1,n}(q,p) - an\mathcal{G}_{m,n-1}(q,p) , \\
 \frac{\partial}{\partial q}\mathcal{G}_{m,n}(q,p) &= m\mathcal{G}_{m-1,n}(q,p) , \quad \frac{\partial}{\partial p}\mathcal{G}_{m,n}(q,p) = n\mathcal{G}_{m,n-1}(q,p) ,
 \end{aligned} \tag{A.4}$$

which provide the differential equation

$$\left[ -c\frac{\partial^2}{\partial q^2} - a\frac{\partial^2}{\partial p^2} + 2b\frac{\partial^2}{\partial q\partial p} + \Delta \left( q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p} \right) \right] \mathcal{G}_{m,n} = \Delta(m+n)\mathcal{G}_{m,n} , \tag{A.5}$$

the same as for  $\mathcal{H}_{m,n}$ .

Finally, let us note that the orthogonality relation (20) in the case we are considering specialize as

$$\int dq \int dp \mathcal{H}_{m,n}(q,p) \mathcal{G}_{r,s}(q,p) \exp \left\{ -\frac{1}{2} \underline{x}^T \underline{\underline{M}} \underline{x} \right\} = \frac{2\pi}{\sqrt{\Delta}} m!n! \delta_{m,r} \delta_{n,s} . \tag{A.6}$$

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